Department of Electrical & Computer Engineering
University of California, Santa Barbara

ECE 240A
Winter 2010
Shynk
H.O. #10

EXAMPLE MIDTERM EXAM

INSTRUCTIONS

This exam is open book and open notes. It consists of 3 problems and is worth a total of 90 points. The problems are not of equal difficulty, so use discretion in allocating your time. Attempt to answer all questions in any order.

1. MEAN-SQUARE ESTIMATION (30 points)

Let $Y = X^3$ where X is a random variable with the following known moments:

$$\mu_m = E[X^m], \quad m = 1,\ 2,\ \ldots$$

For example, the mean of X (which is not necessarily zero) is given by μ_1.

(a) Write expressions for the variance of X and the variance of Y in terms of the known $\{\mu_m\}$.

(b) Find the *nonlinear* mean-square (MS) estimator of Y given X.

(c) Find an expression for the *linear* (affine) MS estimator in terms of the known $\{\mu_m\}$.

2. LEAST-SQUARES ESTIMATION (30 points)

Suppose that we have N independent measurements generated as follows:

$$z(k) = \theta_1 - \theta_2 + v(k)$$

where the $\{\theta_i\}$ are unknown parameters, and $v(k)$ is a white noise process with zero mean and variance σ_v^2. Assume that we have N independent measurements of $z(k)$.

(a) For the parameter vector $\theta = [\theta_1, \theta_2]^T$, write the measurement vector in the standard form for this linear measurement model.

(b) Find the least-squares (LS) estimator for θ. Determine if it is possible to separately estimate θ_1 and θ_2.

(c) Repeat parts (a) and (b) for the following measurements:

$$\begin{aligned} z_1(k) &= \theta_1 - \theta_2 + v_1(k) \\ z_2(k) &= \theta_1 + v_2(k) \end{aligned}$$

where $\mathbf{v}(k) = [v_1(k), v_2(k)]^T$ contains mutually uncorrelated white noise processes, each with zero mean and variance σ_v^2. Assume that we have N independent measurements of the vector $\mathbf{z}(k) = [z_1(k), z_2(k)]^T$.

3. CRAMER-RAO LOWER BOUND (30 points)

(a) Consider the scalar function $g(\theta)$ of the unknown parameter θ. Derive the following Cramer-Rao lower bound (CRLB) for the unbiased estimator $\hat{g}(N)$ of $g(\theta)$ based on N measurements:

$$\text{var}(\hat{g}(N)) \geq \frac{\left(\frac{\partial g(\theta)}{\partial \theta}\right)^2}{E\left[\frac{\partial}{\partial \theta}\ln p(\mathbf{z})\right]^2}$$

where $p(\mathbf{z})$ is the joint probability density function of the measurements $\{z(1), \ldots, z(N)\}$.

(b) Consider the two-sided exponential density with parameter $\lambda > 0$:

$$p(z) = \frac{\lambda}{2} e^{-\lambda |z|}.$$

Determine the CRLB for $\theta = \lambda$ based on N measurements of $z(k)$.

(c) Repeat part (b) for $\theta = 1/\lambda$.