Department of Electrical & Computer Engineering
University of California, Santa Barbara

ECE 240A
Winter 2010
Shynk
H.O. #21

EXAMPLE FINAL EXAM

INSTRUCTIONS

This exam is open book and open notes. It consists of 4 problems and is worth a total of 160 points. The problems are not of equal difficulty, so use discretion in allocating your time. Attempt to answer all questions in any order.

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1. SUFFICIENT STATISTICS AND ML ESTIMATION (40 points)

Suppose that $\{X_1, X_2, \ldots, X_N\}$ is a random sample drawn according to the following probability mass function (pmf):

$$p_X(x) = \left(\begin{array}{c} 2 \\ x \end{array} \right) \theta^x (1-\theta)^{2-x} I_{\{0,1,2\}}(x), \tag{1}$$

where $I(x)$ is the indicator function and $\theta > 0$ is an unknown (nonrandom) parameter.

(a) Find a one-dimensional sufficient statistic.

(b) This statistic is obviously minimal sufficient. Determine if it is also complete. (Hint: Note that the pmf of the sufficient statistic is similar to that in (1), except that it has parameters $\{\theta, 2N\}$.)

(c) Find the maximum-likelihood estimators for $P[X_i = 0] = \theta$ and $P[X_i = 2] = \theta^2$.

(d) Determine if these estimators are unbiased.

2. KALMAN PREDICTOR (40 points)

When the measurement noise $v(k)$ and the unmeasurable disturbance $w(k)$ are correlated (with cross-correlation matrix S), the one-step Kalman predictor is given by

$$\hat{x}(k+1|k) = \Phi\hat{x}(k|k-1) + \Psi u(k) + L(k)\tilde{z}(k|k-1) \tag{2}$$

$$L(k) = [\Phi P(k|k-1)H^T + \Gamma S] \times [HP(k|k-1)H^T + R]^{-1} \tag{3}$$

$$\begin{aligned} P(k+1|k) &= [\Phi - L(k)H]P(k|k-1)[\Phi - L(k)H]^T \\ &\quad + \Gamma Q \Gamma^T - \Gamma S L^T(k) - L(k)S\Gamma^T + L(k)RL^T(k) \end{aligned} \tag{4}$$

where $L(k)$ is the Kalman predictor gain, $\tilde{z}(k|k-1)$ is the usual prediction error, and we have assumed that the state-variable parameters are time invariant. Note that if $S = 0$, these equations reduce to the standard Kalman predictor covered in class.

Consider the following scalar measurable process:

$$z(k) = v(k) + w(k-1) \tag{5}$$

where $v(k) = w(k)$ is a stationary white-noise sequence with zero mean and unit variance.

(a) For this process, determine the parameters of the first-order (scalar) state-variable model, i.e., specify Φ, Ψ, Γ, H, and the variances Q and R.

(b) For the system in (5) and from your answer in part (a), show that the one-step Kalman predictor of $z(k+1)$ is

$$\hat{z}(k+1|k) = L(k)[z(k) - \hat{z}(k|k-1)].$$

(c) For $S = 1$, show that (3) and (4) reduce to

$$L(k) = [P(k|k-1) + 1]^{-1} \quad \text{and} \quad P(k+1|k) = 1 - L(k). \tag{6}$$

(d) Let $B(k) = P^{-1}(k|k-1)$ and show that, with the initial condition $B(0) = 1$, $L(k)$ in (6) is

$$L(k) = \frac{k+1}{k+2}.$$

3. UNBIASED ESTIMATION AND MAP ESTIMATION (40 points)

Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are two independent unbiased estimators of θ with variances σ_1^2 and σ_2^2, respectively. Consider the new estimator

$$\hat{\theta} = \alpha\hat{\theta}_1 + \beta\hat{\theta}_2$$

where α and β are nonrandom constants.

(a) Find α and β such that the variance of $\hat{\theta}$ is minimum and $\hat{\theta}$ is unbiased.

(b) What is the corresponding minimum value of the variance?

(c) What are the optimal values of α and β as $\sigma_2^2 \to \infty$? What is the corresponding minimum value of the variance?

Consider now the random variable Z which has a Poisson distribution with mean θ:

$$f(z) = \frac{\theta^z}{z!} e^{-\theta} I_{\{0,1,\ldots\}}(z)$$

where θ is also a random variable with the following exponential distribution:

$$f(\theta) = \lambda e^{-\lambda\theta} I_{[0,\infty)}(\theta)$$

where the constant parameter $\lambda > 0$ is known. (Note that parts (d) and (e) below are independent of parts (a)–(c) above.)

(d) Find the maximum a posteriori (MAP) estimator of θ from the single measurement Z.

(e) How does you answer compare to the maximum likelihood (ML) estimator of θ?

4. MS ESTIMATION (40 points)

Suppose that X and Y are correlated random variables with the following density functions:

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x-y-\frac{y^2}{2}\right)^2}$$

$$f_Y(y) = e^{-y} I_{[0,\infty)}.$$

Note that since Y is exponential with parameter $\lambda = 1$, the moments are $E(Y^k) = k!$.

(a) Find the nonlinear minimum mean-square (MS) estimator $\hat{X}$ of X given that we observe Y.

(b) From your answer in part (a), determine the corresponding minimum value of the MS error.

(c) Find the linear minimum MS estimator of X given that we observe Y. Your answer should be of the form $\hat{X} = aY + b$ where a and b are constants to be determined.

(d) From your answer in part (c), determine the corresponding minimum value of the MS error. (Hint: Suppose we know $E(U|V)$. Then $E(U) = E_V[E(U|V)]$ where the outer expectation is over V.